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ON THE QUASI-HADAMARD PRODUCT OF CERTAIN UNIVALENT FUNCTIONS

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Abstract. We improve some recent results due to Kumar (J. Math. Anal. Appl. 126 (1987), 70-77) concerning the quasi-Hadamard product of certain starlike and convex univalent functions.

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1. Introduction. Let A denote the family of functions f which are analytic in the unit disk $E = \{ z : |z| < 1 \}$ and normalised by $f(0) = f'(0) - 1 = 0$. Let S denote the subfamily of A consisting of functions that are univalent in E. A function $f \in S$ is in $S^*(\alpha)$, the class of starlike functions of order $\alpha (0 \leq \alpha < 1)$ if and only if $\operatorname{Re}\{zf'(z)/f(z)\} > \alpha$, $z \in E$. Further, $f \in S$ is in $C(\alpha)$, the class of convex functions of order α if and only if $zf'(z) \in S^*(\alpha)$.

Let T denote the subclass of S consisting of functions whose non-zero coefficients, from the second on, are negative; that is, an analytic and univalent function $f \in T$, if and only if it can be expressed in the form

$$(1.1) \qquad f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0.$$

Further, we denote $ST_0^*(\alpha)$ and $C_0^*(\alpha)$, $0 \leq \alpha < 1$, the classes obtained by taking intersections, respectively, of the classes $S^*(\alpha)$ and $C(\alpha)$ with T. These classes were introduced and studied by Silverman [9].

For a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic in E, we define the differential operator D^n, $n \in N_0 = \{0, 1, 2, \dots\}$ by

(i) $D^0 f(z) = f(z)$

(ii) $D^1 f(z) = zf'(z)$

(iii) $D^n f(z) = D(D^{n-1} f(z))$.

This operator was introduced by Salagean [8]. We note that if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in E, then $D^n f(z) = \sum_{k=0}^{\infty} k^n a_k z^k$.

Let $S_n^*(\alpha)$ denote the class of function $f \in T$ such that

$$\mathrm{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \alpha, \quad n \in N_0$$

for $z \in E$ and $0 \leq \alpha < 1$. It is easily seen that $S_0^*(\alpha) \equiv ST_0^*(\alpha)$ and $S_1^*(\alpha) = C_0^*(\alpha)$, $0 \leq \alpha < 1$.

A necessary and sufficient condition for a function f defined by (1.1) to be in $S_n^*(\alpha)$ is that

$$(1.2) \qquad \sum_{k=2}^{\infty} k^n (k-\alpha)\, a_k \leq (1-\alpha).$$

A more general form of this result can be found in [7].

From (1.2), it follows that for any positive integer n

$$S_n^*(\alpha) \subset S_{n-1}^*(\alpha) \subset \cdots \subset S_2^*(\alpha) \subset C_o^*(\alpha) \subset ST_o^*(\alpha)$$

and

$$S_n^*(\alpha_2) \subset S_n^*(\alpha_1), \quad 0 \le \alpha_1 < \alpha_2 < 1.$$

We also note that for every $n \in N_o$; the class $S_n^*(\alpha)$ is non-empty as the functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} k^{-n} \{(1-\alpha)/(k-\alpha)\} \lambda_k z^k,$$

where $0 \le \alpha < 1$, $\lambda_k \ge 0$ and $\sum_{k=2}^{\infty} \lambda_k \le 1$, satisfy the inequality (1.2).

Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$, $a_k \ge 0$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$, $b_k \ge 0$. The quasi-Hadamard product of the functions f(z) and g(z) is defined by

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k.$$

Similarly, we can define the quasi-Hadamard product of more than two functions. We note that Padmanabhan and Manjini [7] used the phrase " Modified Hadamard product" instead of "Quasi-Hadamard product" in this definition.

Problems concerning the quasi-Hadamard product of two or more functions have been considered by many researchers [1,2,3,4,6,7]. Recently, Kumar [2] has established the following theorems for the quasi-Hadamard product.

Theorem A. For each i = 1,2,...,m, let the functions f_i belong to the classes $ST_o^*(\alpha_i)$ $(0 \le \alpha_i < 1)$, respectively.

Then, the quasi-Hadamard product $(f_1 * f_2 * \cdots * f_m)$ belongs to the class $S^*_{m-1}(\alpha^*)$, where $\alpha^* = \max\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$.

<u>Theorem</u> B. For each $i = 1, 2, \ldots, m$, let the functions f_i belong to the classes $C^*_0(\alpha_i)$ $(0 \leq \alpha_i < 1)$, respectively. Then, the quasi-Hadamard product $(f_1 * f_2 * \cdots * f_m)$ belongs to the class $S^*_{2m-1}(\alpha^*)$, where $\alpha^* = \max\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$.

<u>Theorem</u> C. For each $i = 1, 2, \ldots, m$, let the functions f_i belong to the classes $ST^*_0(\alpha_i)$, respectively; and for each $j = 1, 2, \ldots, q$, let the functions g_j belong to the classes $C^*_0(\beta_j)$ $(0 \leq \beta_j < 1)$, respectively. Then, the quasi-Hadamard product $(f_1 * f_2 * \cdots * f_m)*(g_1 * g_2 * \cdots * g_q)$ belongs to the class $S^*_{m+2q-1}(\gamma)$, where $\gamma = \max\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_q\}$

<u>Theorem</u> D. For each $i = 1, 2, \ldots, m$, let the functions f_i belong to the class $C^*_0(\alpha)$, and let $0 \leq \alpha \leq r_0$, where r_0 is a root of the equation $2^m(1 - mr) - (1 - r)^m = 0$ in the interval $(0, \frac{1}{m})$. Then, the quasi-Hadamard product $f_1 * f_2 * \cdots * f_m$ belongs to the class $S^*_{m-1}(m\alpha)$.

The object of the present paper is to improve Theorems A, B, C and D by using a different technique. The classes, to which the quasi-Hadamard product belongs, determined by us are smaller than those given by Kumar [2]. Evidently, our results are more inclusive as well as applicable, and thus improve theorems A, B, C and D.

Unless otherwise mentioned, we assume throughout this paper that the functions of the form

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i}\, z^k, \quad a_{k,i} \geq 0$$

and

$$g_j(z) = z - \sum_{k=2}^{\infty} b_{k,j}\, z^k, \; b_{k,j} \geq 0,$$

are analytic in the unit disc E. We, further, assume that $0 \leq \alpha_i < 1$, $0 \leq \beta_j < 1$ and $n_i \in N_o = \{0,1,2,\ldots\}$.

2. Main Results

First, we prove

Theorem 1. Let the functions f_i be in $S^*_{n_i}(\alpha_i)$ for each $i = 1,2$, respectively. Then, the quasi-Hadamard product $f_1 * f_2$ belongs to $S^*_p(\gamma)$, where $p = n_1 + n_2 + 1$ and

$$(2.1) \qquad \gamma \equiv \gamma(\alpha_1, \alpha_2) = \frac{2(\alpha_1+\alpha_2) - 3\alpha_1\alpha_2}{2 - \alpha_1\alpha_2}.$$

The result is best possible.

Proof: In view of (1.2), it is sufficient to prove that

$$\sum_{k=2}^{\infty} k^{n_1+n_2+1} (k-\gamma)\, a_{k,1}\cdot a_{k,2} \leq (1-\gamma).$$

Since $f_i \in S_{n_i}(\alpha_i)$ for $i = 1,2$, we have

$$\sum_{k=2}^{\infty} k^{n_i}(k-\alpha_i)a_{k,i} \leq (1-\alpha_i).$$

Therefore, by virtue of Cauchy-Schwarz inequality,

$$(2.2) \qquad \sum_{k=2}^{\infty} \left\{k^{n_1+n_2} \frac{(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}\right\}^{1/2} \sqrt{a_{k,1}\cdot a_{k,2}} \leq 1.$$

Thus, we need to find the largest γ such that

$$\sum_{k=2}^{\infty} k^{n_1+n_2+1} \frac{(k-\gamma)}{(1-\gamma)} a_{k,1} \cdot a_{k,2} \le \sum_{k=2}^{\infty} \left\{ k^{n_1+n_2} \frac{(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)} \right\}^{1/2} \sqrt{a_{k,1} \cdot a_{k,2}}$$

or, equivalently, that

$$\sqrt{a_{k,1} \cdot a_{k,2}} \le \left\{ \frac{k^{n_1+n_2}(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)} \right\}^{1/2} \cdot \frac{(1-\gamma)}{k^{n_1+n_2+1}(k-\gamma)}, \quad k \ge 2.$$

In view of (2.2), it is enough to find the largest γ such that

$$\left\{ \frac{(1-\alpha_1)(1-\alpha_2)}{k^{n_1+n_2}(k-\alpha_1)(k-\alpha_2)} \right\}^{1/2} \le \left\{ \frac{k^{n_1+n_2}(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)} \right\}^{1/2} \cdot \frac{(1-\gamma)}{k^{n_1+n_2+1}(k-\gamma)}, \quad k \ge 2.$$

That is,

$$\text{(2.3)} \qquad \gamma \le \frac{(k-\alpha_1)(k-\alpha_2) - k^2(1-\alpha_1)(1-\alpha_2)}{(k-\alpha_1)(k-\alpha_2) - k(1-\alpha_1)(1-\alpha_2)}$$

$$= \frac{k(\alpha_1+\alpha_2) - (k+1)\alpha_1\alpha_2}{k - \alpha_1\alpha_2}, \quad k \ge 2.$$

We denote the right hand side of (2.3) by $\Phi(k)$ and show that $\Phi(k)$ is an increasing function of $k \ge 2$. This will be true if for $k \ge 2$

$$\text{(2.4)} \qquad \Phi(k+1) - \Phi(k) = \frac{(k+1)(\alpha_1+\alpha_2) - (k+2)\alpha_1\alpha_2}{(k+1-\alpha_1\alpha_2)} - \frac{k(\alpha_1+\alpha_2) - (k+1)\alpha_1\alpha_2}{(k-\alpha_1\alpha_2)} > 0.$$

On simplifying (2.4), we get

$$\Phi(k+1) - \Phi(k) = \frac{(1-\alpha_1)(1-\alpha_2)}{(k+1-\alpha_1\alpha_2)(k-\alpha_1\alpha_2)}$$

which is certainly positive for $k \ge 2$ and $0 \le \alpha_1, \alpha_2 < 1$. Thus, (2.4) holds true. Putting $k = 2$ in (2.3), we deduce (2.1).

The result is best possible for the functions of the form

$$f_i(z) = z - \frac{(1-\alpha_i)}{2^{n_i}(2-\alpha_i)} z^2, \quad i = 1,2.$$

The above theorem can be extended for more than two functions which is as follows.

Theorem 2. Let the functions f_i be in $S^*_{n_i}(\alpha_i)$ for each $i = 1,2,\ldots, m$, respectively. Then the quasi-Hadamard product $(f_1 * f_2 * \cdots * f_m)$ belongs to $S^*_p(\gamma_m)$, where $p = n_1+n_2+\ldots+ n_m+ m-1$ and γ_m is given by

$$(2.5) \qquad \gamma_m \equiv \gamma_m(\alpha_1,\alpha_2,\ldots, \alpha_m) = \frac{\prod_{i=1}^{m}(2-\alpha_i) - 2^m \prod_{i=1}^{m}(1-\alpha_i)}{\prod_{i=1}^{m}(2-\alpha_i) - 2^{m-1} \prod_{i=1}^{m}(1-\alpha_i)}.$$

The result is best possible.

Proof: We prove by induction on m. From Theorem 1, it follows that the result is true for $m = 2$. Let us assume that (2.5) is true for $m = s-1$. Then, we shall prove it for $m = s$. By assumption, $(f_1 * f_2 * \cdots * f_{s-1})$ belongs to the class $S^*_{p_o}(\gamma_{s-1})$, where $p_o = n_1 + n_2 + \ldots n_{s-1} + (s-2)$ and γ_{s-1} is given by

$$\gamma_{s-1} = \frac{\prod_{i=1}^{s-1}(2-\alpha_i) - 2^{s-1} \prod_{i=1}^{s-1}(1-\alpha_i)}{\prod_{i=1}^{s-1}(2-\alpha_i) - 2^{s-2} \prod_{i=1}^{s-1}(1-\alpha_i)}.$$

Since $f_s \in S^*_{n_s}(\alpha_s)$, by using Theorem 1, we deduce that the quasi-Hadamard product $(f_1 * f_2 * \cdots * f_{s-1}) * f_s$ belongs to the class $S^*_{p_1}(\gamma_s)$, where $p_1 = p_0 + n_s + 1$ and γ_s is given by

$$\gamma_s = \frac{2(\gamma_{s-1} + \alpha_s) - 3\,\gamma_{s-1}\cdot\alpha_s}{2 - \gamma_{s-1}\cdot\alpha_s}, \tag{2.6}$$

which on simplification yields

$$\gamma_s = \frac{\prod_{i=1}^{s}(2-\alpha_i) - 2^s \prod_{i=1}^{s}(1-\alpha_i)}{\prod_{i=1}^{s}(2-\alpha_i) - 2^{s-1}\prod_{i=1}^{s}(1-\alpha_i)}.$$

This completes the proof of Theorem 2.

It is easy to see that the result is best possible for the functions of the form

$$f_i(z) = z - \frac{2^{-n_i}(1-\alpha_i)}{(2-\alpha_i)} z^2, \quad 1 \le i \le m.$$

<u>Remark</u>. From (2.1), we note that $\gamma \ge \alpha_1$ and $\gamma \ge \alpha_2$. Similarly, from (2.5), it follows that for $i = 1,2,\ldots, m$

$$\gamma_i \ge \alpha_j, \quad j = 1,2,\ldots, i.$$

from which, we have

$$\gamma_i \ge \max\{\alpha_1, \alpha_2, \ldots, \alpha_i\} = \lambda_i \quad \text{(say)}.$$

Thus,

$$S^*_n(\gamma_i) \subseteq S^*_n(\lambda_i)$$

for each $i = 1,2,\ldots, m$ and $n \in N_0$. We, further, note that the containment is proper if $m \ge 2$.

Putting $n_i = 0$ for each $i = 1,2,\ldots,m$ in Theorem 2, we have

Corollary 1. Let the functions f_i be in $ST_0^*(\alpha_i)$ for each $i = 1,2,\ldots,m$, respectively. Then the quasi-Hadamard product $(f_1 * f_2 * \cdots * f_m)$ belongs to $S_{m-1}^*(\gamma_m) \subseteq S_{m-1}^*(\lambda)$, where γ_m is defined as in (2.5) and $\lambda = \max(\alpha_1,\alpha_2,\ldots,\alpha_m)$.

The result is best possible.

Letting $n_i = 1$ for each $i = 1,2,\ldots,m$ in Theorem 2, we have

Corollary 2. Let the functions f_i be in $C_0^*(\alpha_i)$ for each $i = 1,2,\ldots,m$, respectively. Then the quasi-Hadamard product $(f_1 * f_2 * \cdots * f_m)$ belongs to $S_{2m-1}^*(\gamma_m) \subseteq S_{2m-1}^*(\lambda)$, where γ_m is defined as in (2.5) and $\lambda = \max(\alpha_1,\alpha_2,\ldots, \alpha_m)$.

The result is best possible.

Corollary 3. For each $i = 1,2,\ldots,m$, let the functions f_i be in $ST_0^*(\alpha_i)$, respectively; and for each $j = 1,2,\ldots,q$, let the functions g_j be in $C_0^*(\beta_j)$, respectively. Then, the quasi-Hadamard product $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q$ belongs to $S_p^*(\gamma_{m,q}) \subseteq S_p^*(\lambda)$, where $p = m+2q-1$, $\lambda = \max(\alpha_1,\alpha_2,\ldots,\alpha_m,\beta_1, \beta_2,\ldots, \beta_q)$ and $\gamma_{m,q}$ is given by

$$\gamma_{m,q} \equiv \gamma_{m,q}(\alpha_1,\alpha_2,\ldots,\alpha_m,\beta_1,\beta_2,\ldots,\beta_q)$$

$$= \frac{\prod_{i=1}^{m}(2-\alpha_i) \prod_{j=1}^{q} (2-\beta_j) - 2^{m+q} \prod_{i=1}^{m} (1-\alpha_i) \prod_{j=1}^{q} (1-\beta_j)}{\prod_{i=1}^{m} (2-\alpha_i) \prod_{j=1}^{q} (2-\beta_j) - 2^{m+q-1} \prod_{i=1}^{m} (1-\alpha_i) \prod_{j=1}^{q} (1-\beta_j)} .$$

The result is best possible.

The proof of Corollary 3 follows from Corollaries 1 and 2 followed by Theorem 1.

Remark. In view of the remark following Theorem 2, we observe that the Corollaries 1,2 and 3 provide better estimate when compared with Theorems A, B and C.

Theorem 3. For each $i = 1,2,\ldots,m$, let the functions f_i be in $C_0^*(\alpha)$, $0 \le \alpha < 1$. Then the quasi-Hadamard product $(f_1 * f_2 * \cdots * f_m)$ belongs to the class $S_{m-1}^*(\gamma)$, where

$$(2.7) \qquad \gamma \equiv \gamma(m,\alpha) = \frac{2\{(2-\alpha)^m - (1-\alpha)^m\}}{2(2-\alpha)^m - (1-\alpha)^m}.$$

The result is best possible.

Proof: Since $f_i \in C_0^*(\alpha)$ for each $i = 1,2,\ldots,m$, we have

$$\sum_{k=2}^{\infty} k(k-\alpha)a_{k,i} \le (1-\alpha).$$

Therefore,

$$(2.8) \qquad \sum_{k=2}^{\infty} k^m \left(\frac{k-\alpha}{1-\alpha}\right)^m \prod_{i=1}^{m} a_{k,i} \le 1.$$

We have to find the largest $\gamma \equiv \gamma(m,\alpha)$ such that

$$\sum_{k=2}^{\infty} k^{m-1} \left(\frac{k-\gamma}{1-\gamma}\right) \prod_{i=1}^{m} a_{k,i} \le 1.$$

In view of (2.8), the above inequality is satisfied if

$$\frac{k-\gamma}{1-\gamma} \le \frac{k(k-\alpha)^m}{(1-\alpha)^m}, \quad k \ge 2$$

that is, if

$$(2.9) \qquad \gamma \le \frac{k[(k-\alpha)^m - (1-\alpha)^m]}{k(k-\alpha)^m - (1-\alpha)^m}, \quad k \ge 2.$$

We shall prove that the right hand side of (2.9) is an increasing function of $k \geq 2$. This will be true if the function

$$(2.10) \qquad \Phi_m(k) = (k^2-1)(k+1-\alpha)^m - k^2(k-\alpha)^m + (1-\alpha)^m$$

is non-negative for each $k \geq 2$ and $m \geq 1$. Now,

$$(2.11) \qquad \Phi_1(k) = k(k-1) > 0.$$

Also, from the recursive formula

$$\Phi_{m+1}(k) = (k-\alpha)\Phi_m(k) + (k-1) \quad (k+1)(k+1-\alpha)^m-(1-\alpha)^m \quad , \ m=0,1,2,\ldots,$$

we have

$$(2.12) \qquad \Phi_{m+1}(k) > (k-\alpha)\Phi_m(k), \quad k \geq 2.$$

Thus, by using (2.11) and (2.12), we deduce that $\Phi_m(k)$ is non-negative for $k \geq 2$ and $m \geq 1$. Now, by putting $k = 2$ in the right hand side of (2.9), we get the required result. This proves Theorem 3.

The result is best possible for the functions of the form

$$(2.13) \qquad f_i(z) = z - \frac{1-\alpha}{2(2-\alpha)} z^2, \ i = 1,2,\ldots, m.$$

Taking $m = 1$ in Theorem 3, we get the following comparable result due to Silverman [9].

Corollary 4. For $0 \leq \alpha < 1$, we have

$$C_0^*(\alpha) \subset ST_0^*\left(\frac{2}{3-\alpha}\right).$$

The result is best possible.

Theorem 4. For each $i = 1,2,\ldots, m$, let the functions f_i belong to the class $C_0^*(\alpha)$, and let $0 \le \alpha \le r_0$, where r_0 is the root of the equation $2^m(1-mr)-(1-r)^m = 0$ in $(0, \frac{1}{m})$. Then, the quasi-Hadamard product $(f_1 * f_2 * \cdots * f_m)$ belongs to the class $S_{m-1}^*(\gamma) \subseteq S_{m-1}^*(\alpha m)$, where γ is defined as in (2.7).

The result is best possible.

Proof : The first half of the theorem, that is; the quasi-Hadamard product $(f_1 * f_2 * \cdots * f_m)$ belongs to the class $S_{m-1}^*(\gamma)$ follows from Theorem 3. It remains to show that

$$S_{m-1}^*(\gamma) \subseteq S_{m-1}^*(m\alpha),$$

where $m \ge 1$, $m\alpha < 1$ and γ is defined as in (2.7). This will be true if

$$2\left\{(2-\alpha)^m - (1-\alpha)^m\right\} \ge m\alpha\left\{2(2-\alpha)^m - (1-\alpha)^m\right\},$$

or, equivalently, if

$$2(1-m\alpha)(2-\alpha)^m - (2-m\alpha)(1-\alpha)^m \ge 0.$$

Since

$$(2-\alpha)^m \ge 2^{m-1}(2-m\alpha) \quad (m \ge 1,\ m\alpha < 1,\ 0 \le \alpha < 1),$$

we have

$$\begin{aligned} &2(1-m\alpha)(2-\alpha)^m - (2-m\alpha)(1-\alpha)^m \\ &\ge 2(1-m\alpha)(2-\alpha)^m - \frac{(2-\alpha)^m(1-\alpha)^m}{2^{m-1}} \\ &= \frac{(2-\alpha)^m}{2^{m-1}}\left\{2^m(1-m\alpha) - (1-\alpha)^m\right\} \ge 0 \end{aligned}$$

for $0 \leq \alpha \leq r_0$, where r_0 is the root of the equation $2^m(1-mr) - (1-r)^m = 0$.

This proves Theorem 4.

The result is best possible for the functions f_i defined by (2.13).

Remark. We observe that Theorem 4 improves Theorem D of Kumar [2].

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